GEOMETRY OF THE MODULI OF HIGHER SPIN CURVES

TYLER J. JARVIS

ABSTRACT. This article treats various aspects of the geometry of the moduli $\mathcal{S}_g^{1/r}$ of r-spin curves and its compactification $\overline{\mathcal{S}}_g^{1/r}$. Generalized spin curves, or r-spin curves, are a natural generalization of 2-spin curves (algebraic curves with a theta-characteristic), and have been of interest lately because of the similarities between the intersection theory of these moduli spaces and that of the moduli of stable maps. In particular, these spaces are the subject of a remarkable conjecture of E. Witten relating their intersection theory to the Gelfand-Dikii (KdV_r) heirarchy. There is also a W-algebra conjecture for these spaces [16] generalizing the the Virasoro conjecture of quantum cohomology.

For any line bundle $\mathcal{K}$ on the universal curve over the stack of stable curves, there is a smooth stack $\mathfrak{S}_{g,n}^{1/r}(\mathcal{K})$ of triples $(X, \mathcal{L}, b)$ of a smooth curve X, a line bundle $\mathcal{L}$ on X, and an isomorphism $b : \mathcal{L}^{\otimes r} \longrightarrow \mathcal{K}$. In the special case that $\mathcal{K} = \omega$ is the relative dualizing sheaf, then $\mathfrak{S}_{g,n}^{1/r}(\mathcal{K})$ is the stack $\mathfrak{S}_{g,n}^{1/r}$ of r-spin curves.

We construct a smooth compactification $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$ of the stack $\mathfrak{S}_{g,n}^{1/r}(\mathcal{K})$, describe the geometric meaning of its points, and prove that it is projective.

We also prove that when r is odd and $g > 1$, the compactified stack of spin curves $\overline{\mathfrak{S}}_g^{1/r}$ and its coarse moduli space $\overline{\mathcal{S}}_g^{1/r}$ are irreducible, and when r is even and $g > 1$, $\overline{\mathcal{S}}_g^{1/r}$ is the disjoint union of two irreducible components. We give similar results for n-pointed spin curves, as required for Witten's conjecture, and also generalize to the n-pointed case the classical fact that when $g = 1$, $\overline{\mathcal{S}}_1^{1/r}$ is the disjoint union of $d(r)$ components, where $d(r)$ is the number of positive divisors of r. These irreducibility properties are important in the study of the Picard group of $\overline{\mathfrak{S}}_g^{1/r}$ [15], and also in the study of the cohomological field theory related to Witten's conjecture [16, 34].

CONTENTS

1. Introduction

In this paper we study the geometry of the stack $\mathfrak{S}_g^{1/r}$ of r-spin curves (smooth algebraic curves X with a line bundle (invertible coherent sheaf) $\mathcal{L}$ and an isomorphism from the rth tensor power $\mathcal{L}^{\otimes r}$ to the cotangent bundle ω_X), as well as the coarse moduli $\mathcal{S}_g^{1/r}$, and also the compactification $\overline{\mathfrak{S}}_g^{1/r}$ of $\mathfrak{S}_g^{1/r}$ by *coherent nets of rth roots* over the stack of stable curves $\overline{\mathfrak{M}}_{g,n}$.

Spin curves provide a finite cover of the moduli of curves, distinct (in the case of $g > 1$) from the moduli of curves with level-r structure. Nor is it the same as the non-Abelian level structures that have been studied recently by Looijenga [23] and Pikaart-de Jong [27]. In the case of $g = n = 1$, however, this space is a disjoint union of the modular curves $Y_1(d)$ for all d dividing r.

This moduli space has many similarities to the moduli of stable maps, including the existence of classes analogous to Gromov-Witten classes and an associated cohomological field theory [16]. But it is not the moduli of stable maps into any variety [16, §5.1]. It can, however, be described as a closed and open substack of the stack of stable maps of balanced twisted curves in the sense of Abramovich and Vistoli [2] into a Deligne-Mumford stack (see [1]).

These moduli spaces are especially interesting because of a conjecture of E. Witten, first described in [33, 34], relating the intersection theory on the (compactified) moduli space of r-spin curves and Gelfand-Dikii hierarchies of order r. This conjecture is a generalization of an earlier conjecture of his, which was proved by Kontsevich (see [20, 21] and [22]). As in the case of Gromov-Witten theory, one can construct a potential function from the intersection numbers of a cohomology class c^{virt} with the tautological ψ classes associated to the universal curve. Witten conjectures that the potential of the theory corresponds to the tau-function of the order-r Gelfand-Dikii (KdV_r) hierarchy. In genus zero, the Witten conjecture is true [16], but in higher genus it remains unproven.

Construction of the moduli stack $\mathfrak{S}_g^{1/r}$, and its compactification $\overline{\mathfrak{S}}_g^{1/r}$, was done in [6] for $r = 2$. For $r \geq 2$ several compactifications were constructed in [14]. Unfortunately, none of those compactifications is smooth for general r, although the space $\text{PURE}_{r,g}$ of [14] is smooth when r is prime. This article has two main goals: first, to construct a geometrically meaningful compactification which is smooth for all r, and second, to describe the irreducible and connected components of this compactification.

1.1. **Overview and Outline of the Paper.**

1.1.1. *Construction of the Compactification.* For any line bundle $\mathcal{K}$ on the universal curve $\mathfrak{C}$ over the stack $\overline{\mathfrak{M}}_g$ of stable curves we will denote by $\mathfrak{S}_g^{1/r}(\mathcal{K})$ the stack of triples $(X, \mathcal{L}, b)$, where X is a smooth curve of genus g, $\mathcal{L}$ is a line bundle on

X, and b is an isomorphism $b : \mathcal{L}^{\otimes r} \longrightarrow \mathcal{K}$. Thus the stack $\mathfrak{S}_g^{1/r}$ of smooth spin curves is simply $\mathfrak{S}_g^{1/r}(\omega)$. We will compactify the stack $\mathfrak{S}_g^{1/r}(\mathcal{K})$ for all r, g, and $\mathcal{K}$, by considering the stack of *coherent nets of rth roots*, as given in Definition 2.3.4. Although the definitions of families of these nets and the proofs of smoothness and other properties are relatively technical, the intuitive ideas are straightforward. Moreover, for many applications (for example, those in [15, 16]) one only needs to know the definitions for geometric points of the stack, which are much simpler, the structure of the universal deformation, as described in Theorem 2.4.2, and the description of the irreducible components, as given in Theorem 3.3.1.

The basic idea is simply that, up to isomorphism, an rth root $\mathcal{L}$ of a bundle $\mathcal{K}$ completely determines a whole collection (net) of roots $\mathcal{L}^{\otimes d}$ for each d dividing r. On the boundary, the limit of $\mathcal{L}$ is a rank-one, torsion-free sheaf, but the degeneration of $\mathcal{L}$ is no longer enough to completely specify a collection of dth roots, if r is not prime, so that structure must be made explicit. Moreover, since there are generally too many torsion-free sheaves in the limit of a degenerating invertible sheaf, we also need to rigidify the structure by explicitly specifying the isomorphisms $\mathcal{L}^{\otimes r} \overset{\sim}{\longrightarrow} \omega_X$ and $(\mathcal{L}^{\otimes d})^{\otimes e} \overset{\sim}{\longrightarrow} \mathcal{L}^{\otimes de}$ (or comparable homomorphisms which are almost isomorphisms over a non-smooth curve).

Thus the geometric objects of interest will be a collection $\{\mathcal{E}_d, c_{d,d'}\}$, called a net, of a rank-one torsion-free sheaf $\mathcal{E}_d$, for each d dividing r, and a homomorphism $c_{d,d'}$ for each d' dividing d dividing r, such that $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \longrightarrow \mathcal{E}_{d'}$ is an isomorphism except where $\mathcal{E}_d$ is not locally free. $\mathcal{E}_1$ is the sheaf $\mathcal{K}$, and the $c_{d,d'}$ should be compatible with one another in the obvious way (see Definition 2.1.4). To prevent certain anomalies (including a potential failure to be separated), we also require that the length of the cokernel of $c_{d,d'}$ at each singularity be $d/d' - 1$. The details of these nets are described in Section 2.

In the case that the underlying curve is smooth, or in fact, whenever $\mathcal{E}_r$ is an invertible sheaf, then the entire structure is canonically determined by $\mathcal{E}_r$ and the isomorphism $c_{r,1} : \mathcal{E}_r^{\otimes r} \longrightarrow \mathcal{K}$. Obviously, when r is prime this collection consists only of the data of $\mathcal{E}_r$ and $c_{r,1} : \mathcal{E}_r^{\otimes r} \longrightarrow \mathcal{K}$.

When the homomorphism $c_{r,1}$ fails to be an isomorphism at some point $\mathfrak{p}$, the completion of the local ring of X at $\mathfrak{p}$ is of the form $\hat{\mathcal{O}}_{X,\mathfrak{p}} \cong A = k[[x,y]]/xy$, and $\mathcal{E}_r$ corresponds to an A-module $E \cong\ <\zeta_1, \zeta_2 | y\zeta_1 = x\zeta_2 = 0>$; and $c_{r,1}$ corresponds to the homomorphism $\zeta_1^r \mapsto x^u$, $\zeta_2^r \mapsto y^v$, for some u and v, with $\zeta_1^i \zeta_2^{r-1} \mapsto 0$ if $0 < i < r$. The restriction on the length of the cokernel amounts to saying that $u + v$ must be exactly r. Thus if $\tilde{A} = k[x] \oplus k[y]$ is the normalization of A, the data of E and $c_{r,1}$ over A correspond exactly to the choice of a *free* $\tilde{A}$-module $\tilde{E}$ with an isomorphism $\tilde{E}^{\otimes r} \overset{\sim}{\longrightarrow} (x^u, y^v)\tilde{A}$. So $\mathcal{E}_r$ corresponds to a sheaf $\tilde{\mathcal{E}}_r$ on the normalization $\tilde{X} \overset{\nu}{\longrightarrow} X$ of X at $\mathfrak{p}$ which is locally free, and $c_{r,1}$ corresponds to an isomorphism $\tilde{\mathcal{E}}_r \overset{\sim}{\longrightarrow} \nu^* \omega_X(-up^+ - vp^-)$ where $\nu^{-1}(\mathfrak{p}) = \{p^+, p^-\}$. The choices of u and v which may occur are, of course, controlled in large part by degree considerations. We will call the (unordered) pair $\{u, v\}$ the *order* of $(\mathcal{E}_r, c_{r,1})$ at $\mathfrak{p}$. If $\mathcal{E}_r$ is locally free at $\mathfrak{p}$, then the order is $\{0, 0\}$.

When u and v are relatively prime, the sheaf $\mathcal{E}_r$ and the homomorphism $c_{r,1}$ still completely determine the remainder of the structure $\{\mathcal{E}_d, c_{d,d'}\}$. In fact, for any order $\{u, v\}$ of $(\mathcal{E}_r, c_{r,1})$ the order $\{u_d, v_d\}$ of $(\mathcal{E}_d, c_{d,1})$ is simply $\{u, v\} \pmod{d}$, and the pullback of $\mathcal{E}_d$ to $\tilde{X}$ is $\tilde{\mathcal{E}}_r^{\otimes r/d} \otimes \mathcal{O}(\frac{(u-u_d)}{d}p^+ + \frac{(v-v_d)}{d}p^-)$, with the pullback of $c_{r,d}$

being the obvious inclusion homomorphism. When d does not divide u (and hence v), the order $\{u_d, v_d\}$ is not $\{0,0\}$ and $\mathcal{E}_d$ is not locally free. In this case $(\mathcal{E}_d, c_{d,1})$ and $c_{r,d}$ are completely determined by $(\mathcal{E}_r, c_{r,1})$. But when $\gcd(u,v) = \ell > 1$ the sheaf $\mathcal{E}_\ell$ is determined only up to its restriction to the normalization $\tilde{X}$ of X at $\mathfrak{p}$. An additional gluing datum is necessary (if $\mathfrak{p}$ is a non-separating node of X) to construct $\mathcal{E}_\ell$ from $\mathcal{E}_r$.

Even when the sheaves $\mathcal{E}_d$ are not locally free, to construct a cohomological field theory [16, 34], and in particular to have a composition axiom, as in quantum cohomology, we need not only roots of $\mathcal{K}$, but also roots of $\mathcal{K} \otimes \mathcal{O}(-\sum m_i p_i)$, for marked points p_i of X and integers m_i. Letting $\mathbf{m} = (m_1, \ldots, m_n)$ be the vector of the orders of vanishing at the p_i's, we call these *roots* (or *nets of roots*) *of $\mathcal{K}$ of type* $\mathbf{m}$. We denote the stack of smooth roots of $\mathcal{K}$ of type $\mathbf{m}$ by $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$, its compactification using coherent nets of roots by $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$. When $\mathcal{K} = \omega$ this is the stack of r-spin curves, and is denoted $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$. It can be shown [1] that the structure of such a coherent net of roots is actually equivalent to the data of a single invertible sheaf $\mathcal{L}$ with an isomorphism $\mathcal{L}^{\otimes r} \longrightarrow \mathcal{K}$ on a twisted (stacky) curve $\mathcal{C}$ in the sense of Abramovich and Vistoli [2], and the type $\mathbf{m}$ corresponds to the indices of stackiness at the marked points of $\mathcal{C}$.

Unfortunately, the most general families of such nets of roots, without further restrictions, do not form a smooth stack. The additional conditions we must place on them, although technical to state, amount simply to insisting that only the most natural and best-behaved families will be included in the stack. In Section 2.2 we recall some results from [14] necessary to give these conditions, and in Section 2.3 we describe these additional conditions, which are essentially equivalent to Abramovich and Vistoli's condition that stable maps should be *balanced*.

Once the definitions are in place, it is relatively straightforward to describe the universal deformation and thus show that this stack is smooth. Moreover, there is a natural morphism from this stack $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ to the Deligne-Mumford stacks $\overline{\text{SPIN}}_{r,g}$ and $\text{PURE}_{r,g}$ of [14], making $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ into the normalization of $\overline{\text{SPIN}}_{r,g}$ and $\text{PURE}_{r,g}$, and in particular, $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ (and $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$) is a smooth Deligne-Mumford stack, finite over $\overline{\mathfrak{M}}_{g,n}$. This is described in Section 2.4

1.1.2. *Geometry.* In Section 3 we treat the geometry of the stack of spin curves and its coarse moduli space. In 3.1 we cover basic properties such as projectivity. In 3.2 we discuss relations between the different spaces. Finally, in 3.3 we treat the question of irreducibility. In particular, we first restrict to the complex numbers and use results of Mess [25] and Powell [28] on the Torelli group, and results of Birman [4] on the kernel of the natural homomorphism from the mapping class group $\Gamma_{g,n}$ of genus g surfaces with n punctures to the group $\Gamma_{g,n-1}$ to give a set of generators (finite when $g \neq 2$) for the Torelli group $\mathcal{I}_{g,n}$ for all g and n. This result, and a characterization due to Sipe [31] of r-spin structures on a given curve as splittings of the Gysin sequence for the punctured tangent bundle, allows us to study the monodromy action on the set of r-spin structures of a given curve. When g is greater than one, and r (or any of the m_i) is odd, this action is transitive, and hence the moduli of r-spin curves of type $\mathbf{m}$ is irreducible. It is well-known (see [26] or [6]) that $\overline{\mathcal{S}}_g^{1/2}$ is the disjoint union of two irreducible components. And when r

and all of the m_i are even, there is a natural morphism $[r/2] : \overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{S}}_{g,n}^{1/2,\mathbf{0}}$ described in Section 3.2, which shows that $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ must be the disjoint union of two at least two pieces. But we show that when $g \geq 2$ monodromy acts transitively on the fibres of the morphism $[r/2]$, so $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ is actually the disjoint union of exactly two irreducible components.

The case of $g = 1$ has special arithmetic interest. When $\mathbf{m} = \mathbf{0}$ it is a classical fact that $\overline{\mathcal{S}}_{1,1}^{1/r,\mathbf{0}}$ is the disjoint union of $d(r)$ irreducible components, where $d(r)$ is the number of divisors of r. Again using our description of the Torelli group, we generalize this fact to show that $\overline{\mathcal{S}}_{1,n}^{1/r,\mathbf{m}}$ is the disjoint union of $d_{1,r}(\mathbf{m})$ components, where $d_{1,r}(\mathbf{m})$ is the number of divisors of $\gcd(r, m_1, \ldots, m_n)$.

These results are then used to show that the irreducibility properties hold in any characteristic prime to r. These irreducibility properties are important in the study of the Picard group of $\overline{\mathfrak{S}}_g^{1/r}$ [15], and also in the study of the cohomological field theory related to Witten's conjecture [16, 34].

1.2. **Conventions and Notation.** By a *curve* we mean a reduced, complete, connected, one-dimensional scheme over a field. A *semi-stable curve of genus g* is a curve with only ordinary double points such that $H^1(X, \mathcal{O}_X)$ has dimension g. And an *n-pointed stable curve* is a semi-stable curve X together with an ordered n-tuple of non-singular points $(p_1, \ldots, p_n)$, such that at least three marked points or double points of X lie on every smooth irreducible component of genus 0, and at least one marked point or double point of X lies on every smooth component of genus one. A stable (or semi-stable) curve is a flat, proper morphism $X \to T$ whose geometric fibres X_t are (semi) stable curves. Except where otherwise indicated, r will be a fixed positive integer, and both g and n will be non-negative integers such that $2g - 2 + n > 0$.

By *line bundle* we mean an invertible (locally free of rank one) coherent sheaf. By *canonical sheaf* we mean the relative dualizing sheaf of a family of curves $f : X \to T$, and this sheaf will be denoted $\omega_{X/T}$, or ω_f, or just ω. Note that for a semi-stable curve, the canonical sheaf is a line bundle. When T is Spec k, for an algebraically closed field k, we will also write ω_X for $\omega_{X/T}$.

2. COHERENT NETS OF ROOTS

2.1. **Quasi-roots and Coherent Nets of Quasi-roots.** To begin we need the definition of torsion-free sheaves.

Definition 2.1.1. A *relatively torsion-free sheaf* (or just torsion-free sheaf) on a family of stable or semi-stable curves $f : X \to T$ is a coherent $\mathcal{O}_X$-module $\mathcal{E}$ that is flat over T, such that on each fibre $X_t = X \times_T$ Spec $k(t)$ the induced $\mathcal{E}_t$ has no associated primes of height one.

We will only be concerned with rank-one torsion-free sheaves. Such sheaves are called *admissible* by Alexeev [3] and *sheaves of pure dimension* 1 by Simpson [30]. Of course, on the open set where f is smooth, a torsion-free sheaf is locally free.

Definition 2.1.2. Given an n-pointed nodal curve X/T, a choice of $\mathbf{m} = (m_1, \ldots, m_n)$, and a rank-one, torsion-free sheaf $\mathcal{F}$ on X, an *rth quasi-root of $\mathcal{F}$ of type $\mathbf{m}$* is a pair $(\mathcal{E}, b)$ consisting of a rank-one, torsion-free sheaf $\mathcal{E}$ on X, and an $\mathcal{O}_X$-module homomorphism $c : \mathcal{E}^{\otimes r} \longrightarrow \mathcal{F}(-\sum m_i D_i)$ with the following properties:

1. $r \cdot \deg \mathcal{E} = \deg \mathcal{F} - \sum m_i$;
2. c is an isomorphism on the open subset of X where $\mathcal{E}$ is locally free; and
3. for each point $\mathfrak{p}$ of each fibre X_t of X where $\mathcal{E}$ is not locally free, the length of the cokernel of c at $\mathfrak{p}$ is $r - 1$.

This definition of a quasi-root reduces to that of [14] when the target $\mathcal{F}$ is locally free. Of course, when X is smooth, any quasi-root is locally free and c is simply an isomorphism.

As explained in [14], the last condition is necessary to ensure separatedness of the stack of quasi-roots of a given sheaf $\mathcal{F}$, and is actually a very natural condition. Indeed, as indicated in the introduction, on a nodal curve X over an algebraically closed field, each singularity (point of X where $\mathcal{E}$ is not locally free) of a quasi-root $(\mathcal{E}, c)$ uniquely determines two positive integers u and v, summing to r: these are the order of vanishing of c on the pullback of $\mathcal{E}$ (mod torsion) to each of the two branches of X through the singularity. In particular, the completion of the local ring of X at the singularity is of the form $\hat{\mathcal{O}}_{X,\mathfrak{p}} \cong A = k[[x,y]]/xy$, and $\mathcal{E}$ corresponds to an A-module $E \cong < \xi_1, \xi_2 | y\xi_1 = x\xi_2 = 0 >$ [29, chap.11, Prop.3]. If $\mathcal{F}$ is locally free, it corresponds to the free A-module A, and c corresponds to the homomorphism $\xi_1^r \mapsto x^u$, $\xi_2^r \mapsto y^v$, and $\xi_1^i \xi_2^{r-i} \mapsto 0$ if $0 < i < r$ [14, Prop. 3.3.1]. If $\mathcal{F}$ is not locally free, then it corresponds to an A-module $F \cong < \zeta_1, \zeta_2 | y\zeta_1 = x\zeta_2 = 0 >$ and c corresponds to the homomorphism $\zeta_1^r \mapsto x^u \zeta_1$, $\zeta_2^r \mapsto y^v \zeta_2$, and $\zeta_1^i \zeta_2^{r-i} \mapsto 0$ for $0 < i < r$. Condition (3) on the cokernel in the definition of a quasi-root is simply the condition that $u + v = r$ when $\mathcal{F}$ is locally free, or $u + v = r - 1$ if $\mathcal{F}$ is not locally free.

Definition 2.1.3. We will call the pair $\{u, v\}$ the *order* of (E, c) at the singularity.

In order to produce a smooth stack, we must consider not just quasi-roots, but also coherent nets of quasi-roots; and we must place some additional conditions on the families of these nets which we will explain in the next section.

Definition 2.1.4. A *type-$\mathbf{m}$, coherent net of r-quasi-roots* of a rank-one torsion-free sheaf $\mathcal{F}$ is a collection $\{\mathcal{E}_d, c_{d,d'}\}$, consisting of a rank-one, torsion-free sheaf $\mathcal{E}_d$ for every divisor d of r, and an $\mathcal{O}_X$-module homomorphism $c_{d,d'} : \mathcal{E}_d^{\otimes d/d'} \longrightarrow \mathcal{E}_{d'}$ for each d' dividing d with the following properties:

1. $\mathcal{E}_1 = \mathcal{F}$ and $c_{1,1} = \mathbf{1}$
2. For each divisor d of r, and for each divisor d' of d let $\mathbf{m}'$ be the n-tuple $(m_1', \ldots, m_n')$ such that m_i' is the unique, non-negative integer, less than d/d', and congruent to $m_i \mod (d/d')$. The homomorphism $c_{d,d'}$ is required to make $(\mathcal{E}_d, c_{d,d'})$ into a d/d' quasi-root of $\mathcal{E}_{d'}$ of type $\mathbf{m}'$.
3. The homomorphisms $\{c_{d,d'}\}$ are compatible:
   For any d'' dividing d' dividing d dividing r, the homomorphism $(c_{d',d''})^{\otimes d'/d''}$ commutes with $c_{d,d''}$ and $c_{d',d''}$; that is the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{E}_d^{\otimes d/d''} & \xrightarrow{\ c_{d,d'}^{\otimes d'/d''}\ } & \mathcal{E}_{d'}^{\otimes d'} \\
& \searrow{\scriptstyle c_{d,d''}} & \Big\downarrow{\scriptstyle c_{d',d''}} \\
& & \mathcal{E}_{d''}
\end{array}
$$

Of course, if $\mathcal{E}_r$ is locally free, then the entire net is completely determined by $(\mathcal{E}_r, c_{r,1})$.

Note 2.1.5. Since the type $\mathbf{m}$ changes with different d's dividing r, a net of roots of $\mathcal{F}$ of type $\mathbf{m}$ is not the same as a net of roots of $\mathcal{F} \otimes \mathcal{O}(-\sum m_i p_i)$ of type $\mathbf{0}$. Still, for any dth root $(\mathcal{E}_d, c_d)$ of $\mathcal{F}$ of type $\mathbf{m}$, there is a uniquely determined dth root $(\mathcal{E}'_d, c'_d)$ of $\mathcal{F}$ of type $\mathbf{m}'$ for every $\mathbf{m}'$ congruent to $\mathbf{m}$ mod d; namely, $\mathcal{E}'_d := \mathcal{E}_d \otimes \mathcal{O}(1/d \sum (m_i - m'_i) p_i)$. Thus the condition in Definition 2.1.4(2), requiring that $\mathbf{m}$ change with the different divisors is not a restriction. Indeed, we would get an isomorphic stack by insisting that all $c_{d,d'}$ be roots of some other type congruent to $\mathbf{m}$ mod d/d'. One advantage to the choice we have given here is that it allows the stack of curves with roots to obey something like the composition rules of quantum cohomology. In particular, near any marked point the structure should behave like a structure obtained by normalizing a node. In our case, normalizing a non-locally-free dth quasi root at a singularity of the root yields a dth quasi root, whose type m^+ and m^- at the normalized points is $u_d - 1$ and $v_d - 1$ where $\{u_d, v_d\}$ is the order of the dth root at the node, and thus the type of the normalized root is bounded between 0 and $d - 2$, inclusive. When the root is locally free, composition rules like those of quantum cohomology still exist for these root nets, as explained in [16], but they are more subtle.

Even considering coherent nets of quasi-roots is not quite sufficient to ensure a smooth stack. To describe the additional structure necessary to make the stack smooth we first recall some results on the local structure of a rank-one, torsion-free sheaf and some results from [14] on the local structure of roots.

2.2. **Background on Local Structures.**

2.2.1. *Local Structure of Torsion-free Sheaves.* Let R be the Henselization of a local ring, of finite type over a field or an excellent Dedekind domain, let $\mathfrak{m}$ be the maximal ideal of R. Every nodal curve $X/\operatorname{Spec} R$ has, at a node, a local ring whose Henselization is isomorphic to A, the Henselization of $R[x, y]/xy - \pi$ at $\mathfrak{m} + (x, y)$ for some $\pi \in \mathfrak{m}$.

Definition 2.2.1. Over the ring A, for each pair $p, q \in R$, such that $pq = \pi$, define $E(p, q)$ to be the A-module generated by two elements ξ_1 and ξ_2, with the relations $x\xi_2 = p\xi_1$, and $y\xi_1 = q\xi_2$.

Theorem 2.2.2 (Faltings[8])**.**   1. *Any torsion-free E of rank one over A is isomorphic to an $E(p, q)$ for some $p, q \in R$ with $pq = \pi$.*
   2. *If p, q, p', q' are all in $\mathfrak{m}$, and $pq = p'q' = \pi$, then $E(p, q)$ is isomorphic to $E(p', q')$ if and only if there exists a unit $\alpha \in R^\times$ such that $p' = \alpha p$, and $q' = \alpha^{-1} q$.*

The condition that E is an rth quasi-root of A implies [14, 5.4.11] that in the above theorem p and q can be assumed to have the property $p^u = wq^v$ in R, for $\{u, v\}$ the order of the root, as described in Definition 2.1.3, and with $w \in R^\times$. Moreover, if w has an rth root in $R^\times$, then w may be assumed to be 1. In particular, there is an étale cover of the base on which w may be assumed to be 1.

2.2.2. *Local Coordinates.* In order to use Faltings' result, we need the definition of a local coordinate, which is really just a way of choosing parameters x, y, and π. From the deformation theory of stable curves, we know that near a singularity $\mathfrak{p}$ of X/T the complete local ring $\hat{\mathcal{O}}_{X,\mathfrak{p}}$ over $\hat{\mathcal{O}}_{T,t}$ is of the form $\hat{\mathcal{O}}_{X,\mathfrak{p}} \cong \hat{\mathcal{O}}_{T,t}[[x,y]]/(xy - \pi)$ for some $\pi \in \hat{\mathcal{O}}_{T,t}$. And over some étale neighborhood T' of t, there is an étale neighborhood U of $\mathfrak{p}$ in $X \times_T T'$ with sections x and y in $\mathcal{O}_U$ such that

1. $xy = \pi \in \mathcal{O}_{T',t}$.
2. The ideal generated by x and y has the discriminant locus of X/T as its associated closed subscheme.
3. The obvious homomorphism $\left(\mathcal{O}_{T',t}[x,y]/(xy - \pi)\right) \to \mathcal{O}_{U,\mathfrak{p}}$ induces an isomorphism on the completions $\left(\hat{\mathcal{O}}_{T',t}[[x,y]]/(xy - \pi)\right) \xrightarrow{\sim} \hat{\mathcal{O}}_{U,\mathfrak{p}}$.

Definition 2.2.3. We call such a system a *local coordinate* for X/T near $\mathfrak{p}$.

Note that a local coordinate is not uniquely determined. It is only determined up to the equivalence relation generated by the following operations:

1. Pullback to étale covers.
2. Change by units: namely $x' = \alpha x, y' = \beta y, \pi' = \sigma \pi$ with $\alpha, \beta \in \mathcal{O}_{X'}^*$, and $\alpha\beta = \sigma \in \mathcal{O}_{T'}^*$.
3. Switching branches; namely, interchanging x and y.

2.2.3. *Local Structure of rth Roots and the Stacks* SPIN, PURE, *and* ROOT. If $E =< \zeta_1, \zeta_2 | p\zeta_1 = x\zeta_2, y\zeta_1 = q\zeta_2 >$ is a rank-one, torsion-free A-module, then any A-module homomorphism from $E^{\otimes r}$ to a torsion-free A-module must necessarily factor through the symmetric product $\mathbf{Sym}^r E$ [14, §3.3]. Thus such a homomorphism can be defined in terms of where it takes the elements

$$\delta_i := \varepsilon_{r-i} := \xi_1^{r-i} \xi_2^i.$$

In [14, §5.4.1] it is shown that for any quasi-root $E^{\otimes r} \xrightarrow{b} A$ there exists an invertible element $a \in A^\times$ such that for any $0 \le i \le u$ the homomorphism b takes δ_i to $ax^{u-i}p^i + \gamma_i$ and for $0 \le j \le v$ the homomorphism b takes ε_i to $ay^{v-j}q^j + \gamma^j$, where γ_i and γ^j are nilpotent elements of A, and are annihilated by π^r.

"Good" quasi-roots are those for which γ_i and γ^j are all zero, and even better are those for which there is some $t \in R$ such that $p = t^v$ and $q = t^u$; or $p = t^{v/c}$ and $q = t^{u/c}$ for $c = \gcd(u,v)$. The former are called simply *roots* in [14] and the latter are called *pure roots*; this usage differs from that in this paper.

The stacks of rth roots with these conditions form the compactifications $\overline{\text{SPIN}}_{r,g}$ and PURE$_{r,g}$, respectively, of the stack of smooth spin curves with an rth root $(\mathcal{E}, b)$ of ω. But neither of these stacks is smooth in general. In particular, if Spec $\mathfrak{o}[[s_1, \ldots, s_{3g-3+n}]]$ is the universal deformation of a curve X, then the universal deformation of a geometric point of $\overline{\text{SPIN}}_{r,g}$ with underlying curve X and with order $\{u_i, v_i\}$ at the node defined by the vanishing of s_i is

Spec $\mathfrak{o}[[P_1, Q_1, \ldots, P_k, Q_k, s_{k+1}, \ldots, s_n, s_{n+1}, \ldots, s_{3g-3+n}]]/(P_i^{u_i} - Q_i^{v_i})$

where the root $\mathcal{E}$ is locally of the form $E(P_i, Q_i)$, and the forgetful morphism $\overline{\text{SPIN}}_{r,g} \longrightarrow \overline{\mathfrak{M}}_g$ is given by $s_i = P_i Q_i$. When $\gcd(u_i, v_i) = 1$ for every i, the universal deformation of a geometric point of PURE$_{r,g}$ is Spec $\mathfrak{o}[[t_1, \ldots, t_k, s_{k+1}, \ldots, s_{3g-3+n}]]$ where $P_i = t_i^{v_i}$ and $Q_i = t_i^{u_i}$. And PURE$_{r,g}$ is clearly the normalization of $\overline{\text{SPIN}}_{r,g}$ in this case. Indeed, PURE$_{r,g}$ is the normalization of $\overline{\text{SPIN}}_{r,g}$ whenever r is prime.

When $(u_i, v_i) > 1$ this no longer holds (See Note 2.4.3)—a fact not made clear in [14].

The most important properties we need to know about $\overline{\text{SPIN}}_{r,g}$ are that it is a Deligne-Mumford stack, such that the forgetful morphism to $\overline{\mathfrak{M}}_g$ is both proper and surjective, and the stack of smooth spin curves forms an open dense substack. Moreover, no part of the proof of these properties depends on special properties of ω, and so for any line bundle $\mathcal{K}$ defined on the universal curve $\mathfrak{C}_{g,n}$ over $\overline{\mathfrak{M}}_{g,n}$ there is a Deligne-Mumford stack $\text{ROOT}_{g,n}^{1/r}(\mathcal{K})$ of curves with "good" rth quasi-roots of $\mathcal{K}$, which is proper and surjects to $\overline{\mathfrak{M}}_{g,n}$. Moreover the universal deformation of a geometric point of $root_{g,n}^{1/r}(\mathcal{K})$ is exactly the same as that for $\overline{\text{SPIN}}_{r,g}$.

2.3. **Power Maps and Roots.** We want to describe the last condition for families of nets of roots necessary to make a smooth stack. Intuitively, this amounts to insisting that the net be completely determined by the rth root $(\mathcal{E}_r, c_{r,d})$, whenever the intermediate roots are not locally free at nodes. We will continue to use the notation of the previous section.

2.3.1. *Power Maps.* To define a general object and homomorphism of the net we will insist, as in the case of $\text{PURE}_{r,g}$, that there be a $\tau \in R$ such that $\tau^v = p$ and $\tau^u = q$ where the order of $\mathcal{E}_r$ at the singularity defined by τ is $\{u\ell, v\ell\}$ for some positive integer ℓ. Then locally $\mathcal{E}_r = E(\tau^v, \tau^u)$ and $\mathcal{E}_d$ should be $E(\tau^{\hat{v}}, \tau^{\hat{u}})$ for some $\hat{u}$ and $\hat{v}$, as explained below.

First we give some notation. Let u, v be two non-negative integers (not necessarily the order of $c_{r,1}$), either both zero or both positive, and let $s = u + v$. Given a local coordinate (U, T', x, y, π) choose an element $\tau \in \mathcal{O}_{T'}$ such that $\tau^s = \pi$. Now for any non-negative integers i and j summing to s, we define

$$E_{i,j} := E(\tau^j, \tau^i) = \; < \xi_1, \xi_2 | \tau^j \xi_1 = x\xi_2, \tau^i \xi_2 = y\xi_1 > .$$

Also, we let $E_{0,0}$ be the free module

$$E_{0,0} := \; < \xi_1, \xi_2 | \xi_1 = \xi_2 > .$$

Then for any $d \geq 1$ let u' and v' be the smallest non-negative integers, congruent (respectively) to du and dv (mod s), and let

$$u'' = \frac{du - u'}{s}$$

and

$$v'' = \frac{dv - v'}{s}.$$

If s divides du (and hence dv), then $u' = v' = 0$ and $u'' = du/s, v'' = dv/s$, so $u'' + v'' = d$. If, however, s does not divide du, then $u' + v' = s$ and $u'' + v'' + 1 = d$. If $u = v = 0$, then $u' = v' = u'' = v'' = 0$.

Definition 2.3.1. Given a local coordinate (U, T', x, y, π), two non-negative integers u and v, either both zero or both positive, and an element $\tau \in \mathcal{O}_{T'}$, such that $\tau^{u+v} = \pi$, as above, define the *dth power of $E_{u,v}$* to be the homomorphism $\phi_d : E_{u,v}^{\otimes d} \longrightarrow E_{u',v'}$ as follows.

Again we let $\delta_i := \varepsilon_{d-i} = \zeta_1^{d-i} \zeta_2^i$ in $\mathbf{Sym}^d E_{u,v}$. The dth power of $E_{u,v}$ is the composition of the canonical map $E_{u,v}^{\otimes d} \longrightarrow \mathbf{Sym}^d E_{u,v}$ with the map

$$\delta_i \mapsto x^{u''-i} \tau^{iv} \zeta_1$$

for $0 \le i \le u''$ and

$$\varepsilon_j \mapsto y^{v''-j}\tau^{ju}\zeta_2$$

for $0 \le j \le v''$, where ζ_1 and ζ_2 are the generators of $E_{u',v'}$, as described above.

To check that this map is well defined we must check that $x\phi_d(\varepsilon_j) = p\phi_d(\varepsilon_{j+1})$, $y\phi_d(\varepsilon_j) = q\phi_d(\varepsilon_{j-1})$, $x\phi_d(\delta_i) = p\phi_d(\delta_{i-1})$, and $y\phi_d(\delta_{i+1}) = q\phi_d(\delta_i)$. When $u' = v' = 0$, we also have potentially duplicate definitions since $\varepsilon_{v''} = \delta_{u''}$, so we must also check that $\phi_d(\varepsilon_{v''}) = \phi_d(\delta_{u''})$. These relations are all easy to check, except in the special cases of $j = v''$ or $i = u''$, where we must check, for example, $x\phi_d(\varepsilon_{v''}) = p\phi_d(\delta_{d-v''-1})$. If $u' = v' = 0$ then this amounts to showing that $x\tau^{v''u}\zeta_2 = px\tau^{(u''-1)v}\zeta_1$, but $p = \tau^v$, and $\zeta_1 = \zeta_2$ in this case, so the equality follows from the fact that $u''v = \frac{duv}{s} = v''u$. If u' and v' are non-zero, then we must show that $x\tau^{v''u}\zeta_2 = p\tau^{u''v}\zeta_1$. This follows from the definition of u'' and v'', as well as the relation $x\zeta_2 = \tau^{v'}\zeta_1$; in particular, $x\tau^{v''u}\zeta_2 = \tau^{v'+v''u}\zeta_1 = \tau^{(sv'+dvu-v'u)/s}\zeta_1 = \tau^{(vv'+dvu)/s}\zeta_1 = \tau^{(v(s-u')+duv)s}\zeta_1 = \tau^{v+(du-u')v/s}\zeta_1 = \tau^{v+u''v}\zeta_1 = p\tau^{u''v}\zeta_1$. Thus we have a well-defined homomorphism $\phi_d : E_{u,v}^{\otimes d} \longrightarrow E_{u',v'}$. Note that when $u = v = 0$ the map ϕ_d is just the canonical isomorphism $E_{0,0}^{\otimes d} \cong A^{\otimes d} \overset{\sim}{\longrightarrow} A \cong E_{0,0}$ given by $\zeta^d \mapsto \zeta$.

When τ is zero, the dth power map reduces to just $\delta_0 \mapsto x^{u''}\zeta_1$, and $\varepsilon_0 \mapsto y^{v''}\zeta_2$, and all other δ_i and ε_i for $0 < i, j < r$ map to zero. Thus the length of the cokernel of ϕ_d at the singularity of the central fibre is just $u'' + v'' = d-1$ when the target is not free (i.e. u' and v' not zero and $\zeta_1 \ne \zeta_2$), and it is $u'' + v'' - 1 = d-1$ when the target is free ($u' = v' = 0$ and $\zeta_1 = \zeta_2$). Consequently, the dth power map is always a special case of a local dth quasi-root of $E_{u',v'}$ of order $\{u'', v''\}$. In particular, if $\mathcal{E}_r$ is locally isomorphic to $E_{i,j}$ for $i + j = s$ and $s|r$, the rth power map takes $E_{i,j}$ to $E_{0,0} = A$ and has order $\{ri/s, rj/s\}$.

Note 2.3.2. Given any two non-negative integers i and j, and any positive integer d, it is easy to see that $E_{i,j}$ has a dth root exactly when the equivalences $dx \equiv i \mod (i + j)$ and $dy \equiv j \mod (i + j)$ can be solved for positive integers x and y. And if d and $i + j$ are relatively prime, then a unique root always exists.

2.3.2. *Roots and Nets of Roots.*

Definition 2.3.3. An *rth root* of a rank-one, torsion-free sheaf $\mathcal{F}$ on X/T is an rth quasi-root $(\mathcal{E}, b)$ of $\mathcal{F}$ such that for each singularity $\mathfrak{p}$ of any fibre X_t/t of X/T, if $\mathcal{E}$ is not locally free at $\mathfrak{p}$, then there is a local coordinate (U, T', x, y, π), and a choice of τ in $\mathcal{O}_{T'}$, such that if $\{\tilde{u}, \tilde{v}\}$ is the order of b at $\mathfrak{p}$ and if $\ell = \gcd(\tilde{u}, \tilde{v})$, then letting $u = \tilde{u}/\ell$ and $v = \tilde{v}/\ell$, we have that $\mathcal{E}$ restricted to U is isomorphic to $E_{u,v}$ and $\mathcal{F}$ restricted to U is isomorphic to $E_{u',v'}$, and such that the induced homomorphism $\tilde{b} : E_{u,v}^{\otimes r} \longrightarrow E_{u',v'}$ is an rth power map with respect to τ.

Of course if T is the spectrum of a field, then $\pi = \tau = 0$ and every quasi-root is a root. Thus the difference between roots and quasi-roots is in the way they vary in families.

It is also worth noting that the stack of stable curves with rth roots of ω is exactly the stack $\textsc{Pure}_{r,g}$ of [14].

Definition 2.3.4. A *coherent net of roots* for a torsion-free sheaf $\mathcal{F}$ on a nodal curve X/T is a coherent net of quasi-roots $\{\mathcal{E}_d, c_{d,d'}\}$ with the property that for each singularity $\mathfrak{p}$ of any fibre of X/T, if $\mathcal{E}_r$ is not locally free, then there is a local

coordinate (U, T', x, y, π) and a choice of τ in $\mathcal{O}_{T'}$, such that if $\{\tilde{u}, \tilde{v}\}$ is the order of $\mathcal{E}_r$ at $\mathfrak{p}$ and $\ell = \gcd(\tilde{u}, \tilde{v})$ then $\mathcal{E}_r$ is isomorphic to $E(\tau^v, \tau^u)$, with $u = \tilde{u}/\ell$, $v = \tilde{v}/\ell$ and every $c_{d,d'}$ is a d/d'-power map with respect to the same parameter τ.

Again, in the special case that T is the spectrum of a field, every coherent net of quasi-roots is a coherent net of roots. If r is prime, then a coherent rth-root net is simply an rth root. Moreover, if $\mathcal{E}_d$ is locally free, then $\mathcal{E}_d$ uniquely determines all $\mathcal{E}_{d'}$ and all $c_{d,d'}$ (up to isomorphism) such that $d'|d$.

Definition 2.3.5. An *isomorphism of two coherent root nets* $\{\mathcal{E}_d, c_{d,d'}\}$ and $\{\mathcal{E}'_d, c'_{d,d'}\}$ is a set of isomorphisms $\{\alpha_d : \mathcal{E}_d \longrightarrow \mathcal{E}'_d\}$ which commute with the homomorphisms $c_{d,d'}$ and $c'_{d,d'}$.

Proposition 2.3.6. *The properties of being an rth root or of being a coherent net of roots is independent of the choice of local coordinate.*

Proof. It suffices to show that for any $\mathcal{E}^{\otimes d} \xrightarrow{c} \mathcal{F}$ for which there exists a local coordinate (U, T', x, y, π) and an element $\tau \in \mathcal{O}_{T'}$ with respect to which $\mathcal{E} \cong E(\tau^v, \tau^u)$ and $\mathcal{F} \cong E(\tau^{v'}, \tau^{u'})$ and c is a dth power map, and given any other local coordinate $(\tilde{U}, \tilde{T}', \tilde{x}, \tilde{y}, \tilde{\pi})$, then there exists an étale neighborhood T'' of t in $T' \times_T \tilde{T}'$ and an étale neighborhood U'' of $\mathfrak{p}$ in $U \times_X \tilde{U}$ over T'' and a choice of $\tilde{\tau}$, such that with respect to the local coordinate $(U'', T'', \tilde{x}, \tilde{y}, \tilde{\pi})$, the sheaf $\mathcal{E}$ is isomorphic to $E(\tilde{\tau}^v, \tilde{\tau}^u)$, the sheaf $\mathcal{F}$ is isomorphic to $E(\tilde{\tau}^{v'}, \tilde{\tau}^{u'})$, and c is a dth power map with respect to $\tilde{\tau}$.

After pulling back to an appropriate étale neighborhood U'' over some étale $T'' \longrightarrow T' \times_T \tilde{T}'$, and after possibly switching branches (interchanging x and y), we may assume that $\tilde{x} = \alpha x$ and $\tilde{y} = \beta y$ for units $\alpha, \beta \in \mathcal{O}_{U''}$, and $\tilde{\pi} = \sigma \pi$ with $\alpha\beta = \sigma \in \mathcal{O}^*_{T''}$.

If $u + v = s$, then let a and b in $\mathcal{O}^*_{U''}$ be sth roots of α and β respectively. Then $E(\tau^v, \tau^u)$, with respect to the coordinate (x, y, π) is $< \xi_1, \xi_2 | x\xi_2 = \tau^v \xi_1, y\xi_2 = \tau^u \xi_2 >$, which is isomorphic to $< \tilde{\xi}_1, \tilde{\xi}_2 | \tilde{x}\tilde{\xi}_2 = (ab\tau)^v \tilde{\xi}_1, \tilde{y}\tilde{\xi}_1 = (ab\tau)^u \tilde{\xi}_2 >$, where $\tilde{\xi} = a^u \xi_1$ and $\tilde{\xi}_2 = b^v \xi_2$. Thus we may take $\tilde{\tau}$ to be $ab\tau$, and $\mathcal{E}$ is isomorphic to $E(\tilde{\tau}^v, \tilde{\tau}^u)$ with respect to the new coordinates $(\tilde{x}, \tilde{y}, \tilde{\pi})$.

Similarly, $\mathcal{F}$, which is isomorphic to $E(\tau^{v'}, \tau^{u'})$ with respect to the old coordinates is $< \zeta_1, \zeta_2 | x\zeta_2 = \tau^{v'} \zeta_1, y\zeta_1 = \tau^{u'} \zeta_2 >$ which is isomorphic to $< \tilde{\zeta}_1, \tilde{\zeta}_2 | \tilde{x}\tilde{\zeta}_2 = \tilde{\tau}^{v'} \tilde{\zeta}_1, \tilde{y}\tilde{\zeta}_1 = \tilde{\tau}^{u'} \tilde{\zeta}_2 >$, where $\tilde{\zeta}_1 = a^{u'} \zeta_1$, and $\tilde{\zeta}_2 = b^{v'} \zeta_2$. Thus, $\mathcal{F}$ is isomorphic to $E(\tilde{\tau}^{v'}, \tilde{\tau}^{u'})$ with respect to the new coordinates $(\tilde{x}, \tilde{y}, \tilde{\pi})$. Moreover, the dth power homomorphism ϕ_d with respect to the old coordinates gives $\delta_i \mapsto x^{u''-1}\tau^{iv}\zeta_1 = \tilde{x}^{u''-i}\tilde{\tau}^{iv}\tilde{\zeta}_1 \alpha^{i-u''}(ab)^{-iv}a^{-u'}$; whereas the dth power homomorphism ψ_d with respect to the new coordinate gives $\delta_i = b^{-iv}a^{-(d-i)u}\tilde{\delta}_i \mapsto b^{-iv}a^{(i-d)n}\tilde{x}^{u''-i}\tilde{\tau}^{iv}\tilde{\zeta}_1$. These agree because $u + v = s$ and $su'' + u' = du$. A similar computation shows that the dth power maps agree on ε_i, and thus the diagram

$$
\begin{array}{ccc}
(E(\tau^v, \tau^u)_{old})^{\otimes d} & \xrightarrow{\phi_d} & E(\tau^{v'}, \tau^{u'})_{old} \\
\Big\downarrow & & \Big\downarrow \\
(E(\tilde{\tau}^v, \tilde{\tau}^u)_{new})^{\otimes d} & \xrightarrow{\psi_d} & E(\tilde{\tau}^{v'}, \tilde{\tau}^{u'})_{new}
\end{array}
$$

commutes. $\square$

Example 2.3.7. When the target sheaf is ω and r is two, then coherent nets on smooth curves correspond to classical spin curves (a curve and a theta-characteristic) *together with an explicit isomorphism* $\mathcal{E}_2^{\otimes 2} \xrightarrow{\sim} \omega$.

Example 2.3.8. If the target sheaf is $\mathcal{O}_X$, then a coherent net of rth roots of $\mathcal{O}$ on a smooth curve X corresponds to an r-torsion point of the Jacobian of X, again with an explicit isomorphism of $\mathcal{E}_r^{\otimes r} \xrightarrow{\sim} \mathcal{O}_X$. In particular, the stack of coherent nets of rth roots of $\mathcal{O}$ on smooth curves of genus one (with one marked point) forms a gerbe over the disjoint union $\coprod_{d|r} Y_1(d)$ of the modular curves $Y_1(d)$ of points of exact order d, where d divides r. Since the stack of nets of roots of $\mathcal{O}$ on stable curves is the normalization with respect to $\overline{\mathfrak{M}}_{g,n}$ of the stack of nets on smooth curves (see Theorem 2.4.4), it forms a gerbe over the union of modular curves $\coprod_{d|r} X_1(d)$.

It is easy to see that if $\mathcal{E}_d$ is locally free, then for each d' dividing d, $\mathcal{E}_{d'}, c_{d',1}$, and $c_{d,d'}$ are all canonically determined (up to isomorphism) as $\mathcal{E}_{d'} = \mathcal{E}_d^{\otimes d/d'}$, $c_{d',1} = c_{d,1}$, and $c_{d,d'} = \mathbf{1}$. Similarly if $\mathcal{E}_{d'}$ is not locally free, then $\mathcal{E}_{d'}$ and $c_{d,d'}$ for each d such that $d'|d|r$ are canonically determined by $\mathcal{E}_r$ and $c_{r,1}$. In particular, off the singularities we have $\mathcal{E}_{d'} = \mathcal{E}_d^{\otimes d/d'} = \mathcal{E}_r^{\otimes r/d'}$, and near the singularities $\mathcal{E}_{d'}$ is determined as a d/d'-power of $\mathcal{E}_d$. Thus étale descent shows that $\mathcal{E}_{d'}$ is globally determined (up to isomorphism) by $\mathcal{E}_r$. And this occurs precisely when d' does not divide $\tilde{u}$ (or $\tilde{v}$), where $\{\tilde{u}, \tilde{v}\}$ is the order of $c_{r,1}$.

In fact, the only additional information we gain from the coherent net of roots that is not inherent in the root $\mathcal{E}_r^{\otimes r} \longrightarrow \mathcal{F}$ is in the case that $\mathcal{F}$ is locally free and $(u, r) = \ell > 1$. In this case, the coherent root net amounts to the choice of a locally free ℓth root $\mathcal{E}_\ell$ of $\mathcal{F}$ and a non-locally-free (r/ℓ)th root of $\mathcal{E}_\ell$.

The actual objects we wish to study are not just coherent root nets on a fixed curve, but rather, fixing an invertible sheaf $\mathcal{K}$ on the universal curve $\mathfrak{C}_{g,n}$ over $\overline{\mathfrak{M}}_{g,n}$, we wish to study pairs $(X/T, \{\mathcal{E}_d, c_{d,d'}\})$ of a curve X/T and a coherent net of rth roots $\{\mathcal{E}_d, c_{c,c'}\}$ of the line bundle $\mathcal{K}|_X$. Of special interest is the case of $\mathcal{K} = \omega_{X/T}$ and its twists by canonical sections.

Definition 2.3.9. A genus-g, n-pointed r-spin curve of type $\mathbf{m}$ is a stable n-pointed curve X/T of genus g and a coherent net of rth roots of $\omega(\mathbf{m}) := \omega_{X/T}(-\sum m_i D_i)$, where D_i is the divisor of X corresponding to the ith section $\sigma_i : T \longrightarrow X$.

Definition 2.3.10. An isomorphism of r-spin curves $(X/T, \{\mathcal{E}_d, c_{d,d'}\})$ and $(X'/T', \{\mathcal{E}_d', c_{d,d'}\})$ is a pair of isomorphisms (i, α) where $i : X/T \longrightarrow X'/T'$ is an isomorphism of stable curves, and $\alpha : \{\mathcal{E}_d, c_{d,d'}\} \longrightarrow \{i^*\mathcal{E}_d, i^*c_{d,d'}\}$ is an isomorphism of coherent nets of roots, compatible with the canonical isomorphism $\omega_{X/T}(\mathbf{m}) \xrightarrow{\sim} i^*\omega_{X'/T'}(\mathbf{m})$.

Example 2.3.11. As explained in example 2.3.8, if $n \geq 1$ and $\mathbf{m} = \mathbf{0}$ then $\omega(\mathbf{m}) = \mathcal{O}$, and a smooth n-pointed r-spin curve of genus 1 is determined by a n-pointed curve of genus 1 with a point of order r on the curve. However, the automorphisms of the underlying curve identify some of these r-spin structures. In particular, when $n = 1$ and r is odd, the elliptic involution acts freely on all the non-trivial r-spin structures, thus there are only $1 + (r^2 - 1)/2$ isomorphism classes of r-spin structures on the generic 1-pointed curve of genus 1.

2.4. **The Stack of Coherent Nets of Roots on Curves.**

Definition 2.4.1. For any g and n such that $2g - 2 + n > 0$, and for any n-tuple of integers $\mathbf{m} = (m_1, \ldots, m_n)$, if $\mathcal{K}$ is a line bundle on the universal curve $stackC_{g,n}$ over $\overline{\mathfrak{M}}_{g,n}$, then we define $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$ to be the stack of genus g, n-pointed, stable curves, together with the data of a coherent net of rth-roots of $\mathcal{K}$ of type $\mathbf{m}$ on the curve. Isomorphisms of these objects are analogous to those described in Definition 2.3.10 (where $\omega(\mathbf{m})$ is replaced by a general line bundle $\mathcal{K}$).

The main result of this section is that $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$ is a smooth Deligne-Mumford stack over $\mathbb{Z}[1/r]$, which is proper (finite) over $\overline{\mathfrak{M}}_{g,n}$.

We begin with some deformation theory. In [14] it is claimed that the universal deformation of a pure spin curve over the spectrum of a field is the r-fold cover of the universal deformation of the underlying curve

$$\text{Spec } \mathfrak{O}[[\tau_1, \ldots, \tau_m, t_{m+1}, \ldots, t_{3g-3}]] \longrightarrow \text{Spec } \mathfrak{O}[[t_1, \ldots, t_{3g-3}]]$$

given by $t_i = \tau_i^r$ when $i \leq m$ (and $\{\tau_i = 0\}$ are the loci of singularities of the underlying curve where the spin structure is not locally free).

This claim is correct for r prime, and even for r composite, provided that no singularity has order $\{u, v\}$ with $\gcd(u, v) > 1$. However, when $\gcd(u, v) > 1$, this is no longer correct, and the additional datum of the coherent net of roots is precisely what is needed to remedy the defect.

Theorem 2.4.2. *Let $\overline{X}$ be a genus-g, stable, n-pointed curve, and let $\mathcal{K}$ be a line bundle defined on the universal universal deformation of the curve $\overline{X}$. Let $\{\overline{\mathcal{E}}_d, \overline{c}_{d,d'}\}$ be a coherent net of rth roots of the restriction $\overline{\mathcal{K}}$ to $\overline{X}$. The universal deformation space of the curve $\overline{X}$ with the net $\{\overline{\mathcal{E}}_d, \overline{c}_{d,d'}\}$ is the cover*

$$\text{Spec } \mathfrak{o}[[\tau_1, \ldots, \tau_m, t_{m+1} \ldots, t_{3g-3+n}]] \longrightarrow \text{Spec } \mathfrak{o}[[t_1, \ldots, t_m, t_{m+1}, \ldots, t_{3g-3+n}]]$$

where $t_i = \tau_i^{r_i}$, and $r_i = r/\gcd(u_i, v_i)$, and $\{u_i, v_i\}$ is the order of the ith singularity of $(\overline{\mathcal{E}}_r, \overline{c}_{r,2})$. In particular, for any $\mathcal{K}$ defined on the universal curve over $\overline{\mathfrak{M}}_{g,n}$, the stack of curves with a coherent net of roots of $\mathcal{K}$ is smooth.

Proof. Let $R = \mathfrak{o}[[\tau_1, \ldots, \tau_m, t_{m+1}, \ldots, t_{3g-3+n}]]$. There exists a unique curve X over Spec R induced from the universal deformation of $\overline{X}$ over $\mathfrak{o}[[t_1, \ldots t_{3g-3+n}]]$. Moreover, on the smooth locus $V \subseteq X$ of $f : X \longrightarrow$ Spec R, there is a unique extension of the net of roots $\{\overline{\mathcal{E}}_d, \overline{c}_{d,d'}\}$ on the special fibre to a net of roots $\{\mathcal{E}_{d,V}, c_{d,d',V}\}$ on V. Similarly, about each singularity of the special fibre, there is a local coordinate with $\pi = t_i$. $(\overline{\mathcal{E}}_r, \overline{c}_{r,1})$ uniquely determine the order $\{\tilde{u}, \tilde{v}\}$, so one may define the obvious coherent root net on the neighborhood U_i of the local coordinate; namely, if $\ell = \gcd(\tilde{u}, \tilde{v})$ and $u = \tilde{u}/\ell$, $v = \tilde{v}/\ell$, and if u', and v' are defined as the smallest non-negative integers congruent mod $(u + v)$ to du and dv, respectively, then let $\mathcal{E}_{d,U_i} := E_{u',v'}$. The homomorphism $c_{d,d'}$ is defined to be the obvious power map. To glue these different nets on V and U_i together requires descent data, in the form of rth roots of the transition functions σ_i of $\mathcal{K}$ on the intersections $U_i \cap V$. On the special fibre these data exist; namely for each d, local isomorphisms $\overline{\mathcal{E}}_d \xrightarrow{\sim} E(0,0)$ uniquely determine descent data on the special fibre as $\overline{\gamma}_{d,i}$ which are dth roots of $\overline{\sigma}_i$.

Now $\overline{\gamma}_{d,i}$ has a unique extension to some $\gamma_{d,i}$ defined on an étale cover of $U_i \cup V$. This is true since it holds on the strict Henselization of the local ring of any point,

and since $U_i \cup V$ is quasi-compact, the small étale site $(U_i \cup V)_{\acute{e}t}$ is Noetherian, and thus there exists an étale cover of $U_i \cup V$ on which $\gamma_{d,i}$ can be defined. The fact that r is invertible implies that the choice of $\gamma_{d,i}$ is unique. Thus we have uniquely determined descent data for $\{\mathcal{E}_{d,V}, c_{d,d',V}\}$ and $\{\mathcal{E}_{d,U_i}, c_{d,d'U_i}\}$ and thus a canonically defined, coherent net $\{\mathcal{E}_d, c_{d,d'}\}$ for $\mathcal{K}$.

For any deformation $(X, \{\mathcal{E}_d, c_{d,d'}\})$ of $(\overline{X}, \{\overline{\mathcal{E}}_d, \overline{c}_{d,d'}\})$ over an Artin local ring S, there is a canonical morphism of Spec S to Spec $\mathfrak{o}[[t_1, \ldots, t_{3g-3+n}]]$ induced by the underlying curve. And for each singularity, (say defined by $t_i = 0$) there is a choice of an element $\alpha_i \in S$ such that $\alpha_i^{r_i} = t_i$, such that the deformation is a net of power maps with respect to α_i. Thus we have a homomorphism $R \longrightarrow S$ (defined by $\tau_i \mapsto \alpha_i$) lifting the map $\mathfrak{o}[[t_i, \ldots, t_{3g-3+n}]] \longrightarrow S$.

The induced net of roots over X/S is locally isomorphic to the given deformation $\{\mathcal{E}_d, c_{d,d'}\}$. But the gluing data induced by this choice of α_i may not be the same as those induced by $\{\mathcal{E}_d\}$. Nevertheless, these gluing data differ at worst by an sth root of unity; and, as explained below, one can replace α_i with $\rho\alpha_i$, for an appropriate sth root of unity ρ, to get a different morphism $R \longrightarrow S$ which will induce not only local isomorphisms of the induced net with $\{\mathcal{E}_d, c_{d,d'}\}$, but which will also induce global isomorphisms.

This shows that the deformation over Spec R is versal. Since the isomorphism functor for coherent nets is unramified (Lemma 2.4.5), the deformation is actually universal. $\qquad\square$

Note 2.4.3. The choice of gluing data is uniquely determined once an isomorphism $\overline{\mathcal{E}}_r \xrightarrow{\sim} E(0,0)$ is chosen at each singularity of the central fibre; however, a different isomorphism can yield potentially different gluing data. The difference between two sets of gluing data induced in this way is root of unity; indeed, any automorphism of $E(0,0)$ compatible with a dth power map to a rank-one free module is easily seen to be of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ with $\alpha^d = \beta^d = 1$, (that is $E(0,0) = <\zeta_1, \zeta_2 | x\zeta_1 = y\zeta_1 = 0 >$ is mapped by $\zeta_1 \mapsto \alpha\zeta_1, \zeta_2 \mapsto \beta\zeta_2$), and such an automorphism induces a change in $\overline{\gamma}$ on the x branch by α and a change on the y branch by β. In particular, if the order of the singularity of $\overline{\mathcal{E}}_r$ is $\{\tilde{u}, \tilde{v}\}$ and if $\gcd(\tilde{u}, \tilde{v}) = \ell$ and $s = r/\ell$, then since $\overline{\mathcal{E}}_r$ has an sth power map to $\overline{\mathcal{E}}_\ell$, which is locally free near the singularity, all automorphisms must induce a change in gluing data which is an sth root of unity.

However, if $v = \tilde{v}/\ell$ and $u = \tilde{u}/\ell$, the automorphism $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ extends to an isomorphism $E(t^v, t^u) \xrightarrow{\sim} E(t^v\alpha/\beta, t^u\beta/\alpha)$. And, since $\gcd(u, v) = 1$, and $u + v = s$, there is a unique sth root of unity ρ such that $\rho^v = \alpha/\beta$ and $\rho^u = \beta/\alpha$. Thus $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ induces an isomorphism $E(t^v, t^u) \xrightarrow{\sim} E(\tilde{t}^v, \tilde{t}^u)$ where $\tilde{t} = \rho t$. Consequently, the different set of gluing data induced by the automorphism $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ could also be induced by making a different choice of the parameter t, but leaving the isomorphism $\overline{\mathcal{E}}_r \xrightarrow{\sim} E(0,0)$ unchanged.

This also illustrates the flaw in the argument of [14] that $\mathrm{PURE}_{r,g}$ has a smooth universal deformation space; in that case there is no sth power map, so an automorphism may induce an rth root of unity which is not an sth root of unity, and

thus cannot be induced by a different choice of t_i. In particular, the deformation given there is not versal.

We can now prove the main theorem of this section.

Theorem 2.4.4. *For any line bundle $\mathcal{K}$ on the universal curve $\mathfrak{C}_{g,n}$ over $\overline{\mathfrak{M}}_{g,n}$, the stack $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$ is the normalization of the stack of "good" quasi-roots $\mathrm{ROOT}_{g,n}^{1/r}(\mathcal{K} \otimes \mathcal{O}(-\sum p_i m_i))$ (see Section 2.2.3), and in particular, it is a smooth, proper Deligne-Mumford stack over $\mathbb{Z}[1/r]$, and the natural forgetful morphism $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \longrightarrow \overline{\mathfrak{M}}_{g,n}$ is finite and surjective.*

Proof. The proof follows directly from the description of the universal deformation and the fact that the diagonal is representable, proper, and unramified (Lemma 2.4.5). In fact, it is easy to produce a scheme T'' which is étale over $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$, since there is a scheme T which is an étale cover of $\mathrm{ROOT}_{g,n}^{1/r}(\mathcal{K} \otimes \mathcal{O}(-\sum m_i p_i))$, and for which there is a local coordinate such that "good" quasi-roots $(\mathcal{E}, b)$ of $\mathcal{K} \otimes \mathcal{O}(-\sum m_i p_i)$ have the form $E(p,q)$ with $p^u = q^v$, and b is defined as $b(\delta_i) = x^{u-i}p^i$ for $0 \le i \le u$ and $b(\varepsilon_j) = y^{v-j}q^j$.

If T'' is the scheme locally defined as $\mathrm{Spec} \prod_{\mu_\ell} \mathcal{O}_T[t]/(t^{v/\ell} - p, t^{u/\ell} - q)$ where $\{u,v\}$ is the order of $(\mathcal{E}, b)$ at the singularity, and $\ell = \gcd(u,v)$, then letting the universal net of rth roots defined on T'' be the obvious one, with the different choices of gluing for $\mathcal{E}_\ell$ indexed by the elements of μ_ℓ, it is clear that T'' is étale over $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}(\mathcal{K})$, and that T'' is the normalization of T. □

Lemma 2.4.5. *The relative (over $\overline{\mathfrak{M}}_{g,n}$) diagonal $\Delta : \overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \times \overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \times_{\overline{\mathfrak{M}}_{g,n}} T \longrightarrow \overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \times_{\overline{\mathfrak{M}}_{g,n}} T$ is representable, proper, and unramified. That is to say, given a stable curve X/T, the functor of isomorphisms of coherent nets of roots of $\mathcal{K}$ on X/T is representable, proper, and unramified.*

Of course, since the diagonal $\overline{\mathfrak{M}}_{g,n} \times \overline{\mathfrak{M}}_{g,n} \longrightarrow \overline{\mathfrak{M}}_{g,n}$ is representable, proper, and unramified [7], this lemma shows that the (absolute) diagonal $\overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \times \overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K}) \longrightarrow \overline{\mathfrak{S}}_{g,n}^{1/r}(\mathcal{K})$ also has those properties.

Proof. The functor of isomorphisms of curves and quasi-roots of a given line bundle $\mathcal{K}$ has all of the stated properties [14, Propositions 4.1.14, 4.1.15, and 4.1.16].[1] But an isomorphism of a coherent net of roots is just a coherent system of isomorphisms of the underlying quasi-roots. Thus the functor of isomorphisms of nets of roots is representable as the locus where the individual isomorphisms of the underlying quasi-roots all agree.

To check properness we use the valuative criterion and check that any isomorphism on the generic fibre extends to the whole curve. Since this holds for the individual isomorphisms of the underlying quasi-roots, the only additional thing to check is that the individual isomorphisms of the terms in the net are all compatible. But it is clear in the case that the base is a discrete valuation ring that any

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[1] Again the proofs there are given only for quasi-roots of the line bundle $\mathcal{K} = \omega$, but they do not depend on any property of ω except the fact that it is a line bundle on the universal curve, thus the results are true for a general $\mathcal{K}$.

isomorphisms that are compatible on the generic fibre must also be compatible on the entire curve.

To check that the functor of isomorphisms is unramified, it suffices to check that any automorphism of a coherent root net over a ring R with square-zero ideal I, such that the automorphism is the identity over $\overline{R} = R/I$ is the identity over R. But again, the automorphisms of the underlying quasi-roots must be trivial on R when they are trivial on $\overline{R}$, and thus the automorphism of the net is also trivial on R.   □

3. Geometry of Spin Curve Stacks and Their Moduli

Our chief interest is in r-spin curves, that is when $\mathcal{K}$ is the canonical (relative dualizing) bundle of the universal curve. We denote the stack of n-pointed, genus-g, stable r-spin curves of type $\mathbf{m}$ by $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$, smooth spin curves by $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ and their coarse moduli spaces by $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ and $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}$, respectively. In this section we study some basic geometric properties of these spaces and the relations between them, and we describe the configuration of their irreducible (and connected) components.

3.1. **Basic Properties.** Since the stacks are all Deligne-Mumford, they have coarse moduli spaces which are *a priori* only algebraic spaces [18, Corollary 1.3.1]. But it is straightforward to see that in our case, the moduli spaces are actually projective schemes.

Proposition 3.1.1. *The moduli spaces* $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ *and* $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}$ *are normal and projective (respectively, quasi-projective).*

Proof. The coarse moduli space of any smooth algebraic stack is normal. (This can be seen from the proof of [32, Proposition 2.8].)

The natural forgetful map $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}$ to the moduli $\overline{\mathcal{M}}_{g,n}$ of stable n-pointed curves is surjective and finite. Thus any ample bundle on $\overline{\mathcal{M}}_{g,n}$ pulls back to an ample bundle on $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ (see [19, proof of 3.11]). But it is well known that $\overline{\mathcal{M}}_{g,n}$ is projective (e.g., [12, Theorem 6.40]), and thus $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ is also.   □

3.2. **Relations Between the Different Spaces.** There are several natural morphisms between the various stacks (and moduli spaces).

1. There is a canonical isomorphism from $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ to $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}'}$ where $\mathbf{m}'$ is n-tuple whose entries are all congruent to $\mathbf{m} \mod r$; namely for any net $\{\mathcal{E}_d, c_{d,d'}\}$ of type $\mathbf{m}$, let $\{\mathcal{E}_d', c_{d,d'}'\}$ be the net given by $\mathcal{E}_d' = \mathcal{E}_d \otimes \mathcal{O}(1/d \sum (m_i - m_i')p_i)$ where p_i is the ith marked point, and $c_{d,d'}'$ is the obvious homomorphism. Because of this canonical isomorphism, we will often assume that all the m_i lie between 0 and $r - 1$ (inclusive).

2. The universal curve over $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ is easily seen to be the stack $\mathfrak{S}_{g,n+1}^{1/r,\mathbf{m}'} \xrightarrow{\pi} \overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$, where $\mathbf{m}'$ is the $(n+1)$-tuple $(m_1, \ldots, m_n, 0)$; and π is the morphism which simply forgets the $(n+1)$st marked point. If m_{n+1} is not congruent to zero $\mod r$, there is no such morphism, since the degree of $\omega(\mathbf{m}')$ is $2g - 2 - \sum_{i=1}^{n+1} m_i$ and the degree of $\omega(\mathbf{m})$ is $2g - 2 - \sum_1^n m_i$, but both cannot be simultaneously divisible by r, and thus at least one stack is empty.

3. If s divides r and $\mathbf{m}'$ is the n-tuple $(m'_1, \ldots, m'_n)$ such that m'_i is the least non-negative integer congruent $m_i \mod s$, then there is a natural map $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}} \xrightarrow{[r/s]}$ $\overline{\mathfrak{S}}_{g,n}^{1/s,\mathbf{m}'}$ defined by forgetting all of the terms in the net except those indexed by divisors of s.

Over a smooth curve, since the nets are determined by $(\mathcal{E}_r, c_{r,1})$ (or $(\mathcal{E}_s, c_{s,1})$ for $\overline{\mathfrak{S}}_{g,n}^{1/s,\mathbf{m}}$), this morphism $[r/s]$ is equivalent to replacing $\mathcal{E}_r$ with $\mathcal{E}_r^{\otimes r/s} \otimes$ $\mathcal{O}(1/s \sum ((m_i - m'_i)p_i)) = \mathcal{E}_s$, and $c_{r,1}$ with $c_{r,1} \otimes i$, where $i : \mathcal{O} \longrightarrow \mathcal{O}(\sum (m_i - m'_i)p_i)$ is the canonical inclusion. In the case that r/s and s are relatively prime, $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ is actually a product

$$\mathfrak{S}_{g,n}^{1/r,\mathbf{m}} \cong \mathfrak{S}_{g,n}^{1/s,\mathbf{m}'} \times_{\mathfrak{M}_{g,n}} \mathfrak{S}_{g,n}^{s/r,\mathbf{m}''}.$$

This follows from the fact that on one hand, the obvious maps $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \mathfrak{S}_{g,n}^{1/s,\mathbf{m}'}$ and $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \mathfrak{S}_{g,n}^{s/r,\mathbf{m}''}$ induce a map from $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ to the product. And on the other hand, the inverse of this map can be constructed as $((X, \mathcal{L}_s, c_s), (X, \mathcal{M}_d, c'_d)) \mapsto$ $(X, \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}, c_s^{ad} \otimes c_d'^{bs})$, where $ad + bs = 1$. It is not difficult to check that this map does not depend on the specific choice of a and b, and so is well-defined.

3.3. Irreducibility.

3.3.1. *Irreducibility over* $\mathbb{C}$. In the special case when r is 2 and $g \geq 1$, it is known (see [6] or [26]) that $\overline{\mathcal{S}}_g^{1/2}$ is the disjoint union of two irreducible components $\overline{\mathcal{S}}_g^{1/2 \text{ even}}$ and $\overline{\mathcal{S}}_g^{1/2 \text{ odd}}$ corresponding to the even and odd theta characteristics, respectively. We now extend these results for general g, r, n, and $\mathbf{m}$, assuming the base field is $\mathbb{C}$.

If $g = 0$, let $\ell_{0,r}(\mathbf{m}) = 1$, if $g = 1$, let $\ell_{1,r}(\mathbf{m})$ be $\gcd(r, m_1, \ldots, m_n)$, and if $g \neq 2$, let $\ell_{g,r}(\mathbf{m})$ be $\gcd(2, r, m_1, \ldots, m_n)$; thus $\ell_{g,r}(\mathbf{m})$ is either 1 or 2 for $g \geq 2$. For any genus g, let $d_{g,r}(\mathbf{m})$ be the number of positive divisors of $\ell_{g,r}(\mathbf{m})$, including $\ell_{g,r}(\mathbf{m})$ and 1.

Theorem 3.3.1. *The moduli space* $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ *is the disjoint union of* $d_{g,r}(\mathbf{m})$ *irreducible components.*

In the case of $g = 0$, $\omega(\mathbf{m})$ has a unique rth root, if any, and so the coarse moduli $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ is either empty or is isomorphic to the coarse moduli space $\overline{\mathcal{M}}_{0,n}$ for all r, n, and $\mathbf{m}$. In either case it is irreducible. Of course the additional structure of the homomorphisms in the net prevents the stack $\mathfrak{S}_{0,n}^{1/r,\mathbf{m}}$ from being isomorphic to the stack $\overline{\mathfrak{M}}_{0,n}$.

Consider now the case that $g \geq 1$. First, to see that the number of irreducible (and connected) components of $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ is at least $d_{g,r}(\mathbf{m})$, consider the morphism

$$[r/\ell_{g,r}(\mathbf{m})] : \overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}} \longrightarrow \overline{\mathcal{S}}_{g,n}^{1/\ell_{g,r},\mathbf{m}'}$$

Since $\ell_{g,r}(\mathbf{m})$ divides all of the m_i, it will be enough to show that $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ has $d_{g,r}(\mathbf{m})$ disjoint irreducible components when r divides all of the m_i. But in this case $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ is canonically isomorphic to $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{0}}$, so it suffices to assume $\mathbf{m} = \mathbf{0}$.

In the case of $g = 1$, ω is trivial, and rth roots of ω are simply r-torsion points of the Jacobian. This case follows from a classical fact.

Lemma 3.3.2. *The moduli space of elliptic curves with an r-torsion point consists of one irreducible connected component for each positive divisor of r.*

Indeed, if $\operatorname{Jac} X$ is represented as the quotient of $\mathbb{C}/\Lambda$ for some lattice $\Lambda = < 1, \tau >$, and $\operatorname{Jac}_r X$ is represented as the quotient $(1/r)\Lambda/\Lambda$, giving an isomorphism $\operatorname{Jac}_r X \xrightarrow{\sim} \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$, then a point corresponding to $(a, b) \in \mathbb{Z}/r\mathbb{Z} \times \mathbb{Z}/r\mathbb{Z}$ is in the irreducible component indexed by the divisor $\gcd(r, a, b)$ [9, V.3.3].

In the case that $g \geq 2$ then $\ell_{g,r}(\mathbf{m})$ is either 1 or 2. In the first case there is nothing to prove, and in the second case the space $\overline{\mathcal{S}}_{g,n}^{1/\ell_{g,r}, \mathbf{0}}$ is $\overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{0}}$. In this case we have the following lemma, which is a generalization of its classical counterpart, and which completes the proof that for all g, r, n, and $\mathbf{m}$ the number of irreducible and connected components is at least $d_{g,r}(\mathbf{m})$.

Lemma 3.3.3. *If m_i is even for every i, then the function $e : \overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{m}} \to \mathbb{Z}/2\mathbb{Z}$, which takes $(X, (\mathcal{E}_2, b))$ to $\dim H^0(X, \mathcal{E}_2) \pmod 2$, is locally constant.*

The proof of Lemma 3.3.3 is an easy generalization of Mumford's proof [26, Section 1] of the corresponding result when $n = 0$. Mumford's idea is to take a divisor D of high degree and make a quadratic form q on $H^0(X, \mathcal{E}_2(D)/\mathcal{E}_2(-D))$. Then we can express $H^0(X, \mathcal{E}_2)$ as the intersection of two maximal q-isotropic subspaces and use the fact [10, pp. 735ff] that the dimension (mod 2) of such an intersection is constant under deformation.

To prove that there are no more irreducible components than $d_{g,r}(\mathbf{m})$ we first note that when $\mathbf{m} \equiv \mathbf{0} \pmod r$, and $g = 1$, the claim holds by Lemma 3.3.2. Similarly, if $r = 2$ and $g \geq 1$ then we have the following lemma.

Lemma 3.3.4. *If m_i is even for every i, and if $\overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{m}\ even}$ and $\overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{m}\ odd}$ are the respective inverse images under e of 0 and 1, then $\overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{m}\ even}$ and $\overline{\mathcal{S}}_{g,n}^{1/2, \mathbf{m}\ odd}$ are irreducible.*

The proof of Lemma 3.3.4 is an easy generalization of Cornalba's proof [6, Lemma 6.3] of the corresponding result when $n = 0$. Cornalba's idea is to check the result explicitly for the cases $g = 1$ and $g = 2$ (i.e., write out all of the square roots of the canonical bundle—or in our case, the square roots of the bundle $\omega(-\sum m_i q_i)$). And then one can use induction and a degeneration argument to reduce to the lower genus case. This argument works just as well for our case, except that in the case of $g = 1$ and $g = 2$, all of the m_i must be even to write out the square roots in the form needed to see that monodromy acts transitively on the even (respectively, odd) square roots.

The proof of Theorem 3.3.1 now follows from the theorem below, which is a generalization of results of Sipe [31] and Hain [11, §13].

Theorem 3.3.5. *For any fixed, smooth curve X of genus $g > 0$, if $\ell_{g,r}(\mathbf{m}) = 1$ then the monodromy group acts transitively on the set $\mathcal{S}_{g,n}^{1/r, \mathbf{m}}[X]$ of isomorphism classes of spin structures on X (where two spin structures that differ by an automorphism of X are not considered to be isomorphic). And if $\ell_{g,r}(\mathbf{m}) > 1$ then the monodromy group acts transitively on each fibre of the map $\mathcal{S}_{g,n}^{1/r, \mathbf{m}}[X] \xrightarrow{[r/\ell_{g,r}(\mathbf{m})]} \overline{\mathcal{S}}_{g,n}^{1/\ell_{g,r}, \mathbf{m}'}[X]$.*

To prove Theorem 3.3.5, we need to further study the action of the mapping class group on the fibres of $\mathcal{S}_{g,n}^{1/r, \mathbf{m}}$ over $\mathcal{M}_{g,n}$. In particular, note that if $\Gamma_{g,n}$ is

the (pure) mapping class group for n-pointed curves, and if $\operatorname{Jac}_r X$ is the group of r-torsion points in the Jacobian of a fixed curve $[X] \in \mathcal{M}_{g,n}$, then the group $\Gamma_{g,n}$ acts on the principal homogeneous $\operatorname{Jac}_r X$-space $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$ in a way compatible with the usual action of $\Gamma_{g,n}$ on $\operatorname{Jac}_r X$.

The monodromy action induces a homomorphism from $\Gamma_{g,n}$ into the group of affine (with respect to $\operatorname{Jac}_r X$), invertible transformations $\mathcal{A}$ of $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$. And we have the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & \Gamma_{g,n} & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Jac}_r X & \xrightarrow{\ T\ } & \mathcal{A} & \longrightarrow & GL_{2g}(\mathbb{Z}/r\mathbb{Z}) & \longrightarrow & 1
\end{array}
$$

Here the horizontal sequence is exact, and the map T simply takes an element η to the automorphism that translates by η. Of course, since the mapping class group preserves the intersection product on $H_1(X, \mathbb{Z})$, the image of $\Gamma_{g,n}$ in $GL_{2g}(\mathbb{Z}/r\mathbb{Z})$ is a subgroup of the symplectic group $SP_{2g}(\mathbb{Z}/r\mathbb{Z})$. And it is well known (see, for example [24, pg. 178]) that the image of $\Gamma_{g,n}$ is all of $SP_{2g}(\mathbb{Z}/r\mathbb{Z})$.

We are especially interested in the elements of $\Gamma_{g,n}$ which act trivially on the homology $H_1(X, \mathbb{Z})$. The subgroup $\mathcal{I}_{g,n}$ of all such elements is called the *Torelli group*. The main step in the proof of Theorem 3.3.5 is the following lemma.

Lemma 3.3.6. *If $g \geq 2$, the image of the Torelli group in the group of translations is the subgroup of $\operatorname{Jac}_r X$ generated by $\{2, m_1, m_2, \ldots, m_n\} \operatorname{Jac}_r X$, and if $g = 1$ then the image of the Torelli group is generated by $\{m_1, \ldots, m_n\} \operatorname{Jac}_r X$. More exactly, if E is the image of $\Gamma_{g,n}$ in $\mathcal{A}$, then we have the following exact sequences.*
For $g \geq 2$

$$
0 \longrightarrow\ <2, m_1, \ldots, m_n> \operatorname{Jac}_r X \longrightarrow E \longrightarrow SP(2g, \mathbb{Z}/r\mathbb{Z}) \longrightarrow 1
$$

and for $g = 1$

$$
0 \longrightarrow\ <m_1, \ldots, m_n> \operatorname{Jac}_r X \longrightarrow E \longrightarrow SP(2, \mathbb{Z}/r\mathbb{Z}) \longrightarrow 1.
$$

The lemma implies Theorem 3.3.5 (transitivity of the monodromy action) in the case that $\ell_{g,r}(\mathbf{m}) = 1$. And on the other hand, if $\ell_{g,r}(\mathbf{m}) > 1$, then the map $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X] \xrightarrow{\ [r/\ell_{g,r}(\mathbf{m})]\ } \mathcal{S}_{g,n}^{1/\ell,\mathbf{m}'}[X]$ is equivariant under the action of the mapping class group, and any two spin structures that map to the same point of $\mathcal{S}_{g,n}^{1/\ell,\mathbf{m}'}[X]$ must differ by a point of $\ell_{g,r}(\mathbf{m}) \cdot \operatorname{Jac}_r X$. Therefore, the orbits of the action of $\ell_{g,r}(\mathbf{m}) \cdot \operatorname{Jac}_r X$, and hence also the orbits of the action of $\Gamma_{g,n}$, are exactly the fibres of this map. Thus all that is necessary for the proof of Theorem 3.3.5 is to prove the lemma. The following proof generalizes and expands ideas from the results of Hain [11, §13] and Sipe [31].

Proof. (of Lemma 3.3.6) Rather than studying the rth roots of the bundle $\omega_X(-\sum m_i p_i)$, it is convenient to dualize and study the action on the rth roots of the bundle $\mathcal{L} := TX(\sum m_i p_i)$, where TX is the tangent bundle to X, Let $\mathcal{L}^o$ denote the S^1-bundle, obtained by removing the zero section from $\mathcal{L}$ and retracting to the unit circle in each fibre. The Euler class of $\mathcal{L}^o$ is $2 - 2g + \sum m_i$ and is divisible by r, so
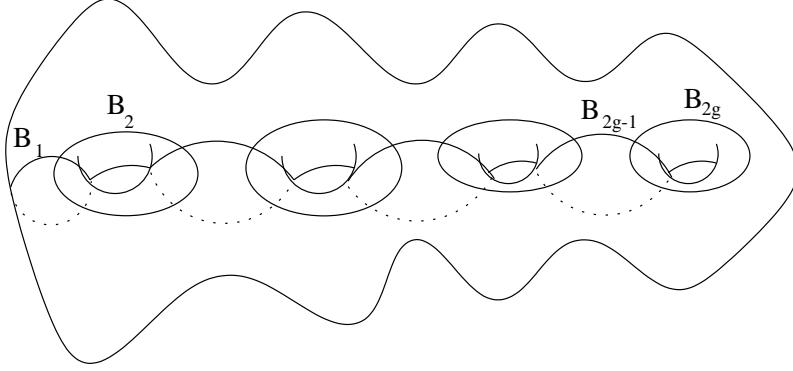
FIGURE 1.   A collection $\{B_1, \ldots, B_{2g}\}$ of cycles forming a basis for $H_1(X, \mathbb{Z})$, such that the intersection form $< B_i, B_{i+1} >= 1$, and the intersection numbers for all other pairs are zero.

the end of the Gysin sequence gives a short exact sequence

$$(1) \qquad 0 \to \mathbb{Z}/r\mathbb{Z} \to H_1(\mathcal{L}^o, \mathbb{Z}/r\mathbb{Z}) \to H_1(X, \mathbb{Z}/r\mathbb{Z}) \to 0.$$

If $\sigma : [0,1] \to X$ is a loop in X that does not pass through any of the n points p_i, then there is a canonical lift of σ to a loop $\tilde{\sigma}$ in $\mathcal{L}^o$, which is defined as $\tilde{\sigma}(t) = \left(\sigma(t), \dfrac{\dot{\sigma}(t)}{||\dot{\sigma}(t)||} \right)$. Note that for any small loop λ, homotopic to zero, which does not pass through any of the p_i, the lift $\tilde{\lambda}$ corresponds to 1 or -1 in the group $\mathbb{Z}/r\mathbb{Z}$ on the left side of the Gysin sequence (1).

Given any rth root $\mathcal{N}$ of $\mathcal{L}$, there is a natural covering map of S^1-bundles $p : \mathcal{N}^o \to \mathcal{L}^o$, which induces a map $\pi_1(\mathcal{L}^o) \to \pi_1(\mathcal{L}^o)/p_*\pi_1(\mathcal{N}^o) \cong \mathbb{Z}/r\mathbb{Z}$, and thus a homomorphism $H_1(\mathcal{L}^o) \to \mathbb{Z}/r\mathbb{Z}$. Moreover, this map takes a lift $\tilde{\lambda}$ of a small loop λ, homotopic to zero in X, and maps it to ± 1. In other words, the rth root induces a splitting of the Gysin sequence. Conversely, given any such splitting, there is a covering space of $\mathcal{L}^o$ that can easily be seen to define an rth root of $\mathcal{L}$. In particular, we have the following fact, originally due to Sipe [31] in the case when $n = 0$.

Fact 3.3.7. *The rth roots of $\mathcal{L}^o$ are in a natural one-to-one correspondence with the splittings of the sequence* (1).

We will compute the action of certain elements of $\Gamma_{g,n}$ on $H_1(\mathcal{L}^o, \mathbb{Z}/r\mathbb{Z})$. Choose a basis for $H_1(\mathcal{L}^o, \mathbb{Z}/r\mathbb{Z})$ in the following way. First take an explicit choice of cycles $B_1, B_2, \ldots, B_{2g}$ which form a basis of $H_1(X, \mathbb{Z}/r\mathbb{Z})$, and such that for all i, j, k, the intersection form $< B_i, B_{i+1} >= 1$, and $< B_j, B_k >= 0$ if k is not $j+1$ or $j-1$. An example of such a collection of cycles is depicted in Figure 1.

We also insist that none of these cycles contain any of the n marked points of X. Now lift these basis cycles in the usual way to $\tilde{B}_1, \ldots \tilde{B}_{2g}$. As before, take a small loop λ, which does not pass through any of the p_i and is homotopic to zero, and lift it to $\tilde{\lambda}$. Now the set $\{\tilde{B}_1, \ldots, \tilde{B}_{2g}, \tilde{\lambda}\}$ gives a basis for $H_1(\mathcal{L}^o, \mathbb{Z}/r\mathbb{Z})$, and it defines a splitting of the sequence (1). Any other splitting is given by simply adding this splitting map $H_1(X) \to H_1(\mathcal{L}^o)$ to an element of $H^1(X, \mathbb{Z}/r\mathbb{Z}) \cong \mathrm{Jac}_r X$. And given a splitting induced by $\{\tilde{B}_1, \ldots, \tilde{B}_{2g}, \tilde{\lambda}\}$, we will index the remaining rth roots by the corresponding 1-cocycle.
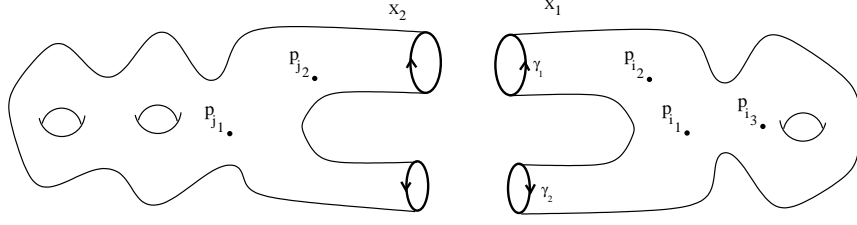
FIGURE 2.   A bounding pair of simple closed curves, separating X into X_1 and X_2 of genus $g_1 = 1$ and $g_2 = 2$ respectively, and containing the marked points $p_{i_1}, \ldots, p_{i_m}$ and the points $p_{j_1}, \ldots, p_{j_{n-m}}$, respectively.

Given any simple closed curve γ on X that misses the marked points $\{p_i\}_{i=1}^n$, there is a corresponding Dehn twist T_γ that is an element of $\Gamma_{g,n}$. Using the Picard-Lefschetz Theorem [35], one can show (see, for example [31, Proposition 3.1]) that the action of T_γ on the lift $\tilde{\sigma}$ to $\mathcal{L}^o$ of a cycle σ on X is given by the following generalized Picard-Lefschetz formula.

$$(2) \qquad\qquad T_\gamma(\tilde{\sigma}) = \tilde{\sigma} + <\sigma, \gamma> \tilde{\gamma}$$

In particular, the action of T_γ on $\tilde{\lambda}$ is trivial.

Now consider a cycle consisting of a bounding pair (sometimes called a *cut pair*) of simple closed curves γ_1 and γ_2 in X which separate X into two surfaces X_1 and X_2 and such that the cycle $\gamma_1 + \gamma_2$ is homologous to zero. Assume these two surfaces are of genus g_1 and g_2, respectively, and assume that the points $p_{i_1}, \ldots, p_{i_m}$ are in X_1, and the remaining points are in X_2 as in Figure 2. If the cycles γ_1 and γ_2 have non-trivial intersection number with B_i, then $<B_i, \gamma_1> = <B_i, \gamma_2>$, and the generalized Picard-Lefschetz formula (2) gives

$$(3) \qquad T_{\gamma_1} \circ T_{\gamma_2}(\tilde{B}_i) = T_{\gamma_2} \circ T_{\gamma_1}(\tilde{B}_i) = \tilde{B}_i + <B_i, \gamma_1> (\tilde{\gamma}_1 + \tilde{\gamma}_2).$$

Since the cycle $\gamma_1 + \gamma_2$ is homologous to zero, the cycle $\tilde{\gamma}_1 + \tilde{\gamma}_2$ in $H_1(\mathcal{L}^o, \mathbb{Z}/r\mathbb{Z})$ is an integral multiple—say $a_{\gamma_1 + \gamma_2}$—of $\tilde{\lambda}$. Thus the action induced by $T_{\gamma_1} \circ T_{\gamma_2}$ on the set of rth roots of $\mathcal{L}$ is simply the one that translates $H^1(X, \mathbb{Z}/r\mathbb{Z})$ by the cocycle that takes B_i to $a_{\gamma_1 + \gamma_2} \cdot <B_i, \gamma_1>$.

We need, therefore, to compute $a_{\gamma_1 + \gamma_2}$. To this end, define a 2-chain in $\mathcal{L}^o$, using, as in [31, Proposition 3.2], a singular unit vector field on X_1. To obtain this vector field, simply glue together $2g_1$ copies of the field represented in Figure 3 to make a vector field on the X_1.

Cutting out a small disc around each point where the vector field is singular, we obtain a surface in the unit tangent bundle TX^o bounded by $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and $2g_1$ cycles homologous to $\tilde{\lambda}$. If we consider the vector field as a section of $\mathcal{L}^o$ instead, singularities at the marked points p_{i_k} do not require the removal of the corresponding discs, and the 2-chain in $\mathcal{L}^o$ is bounded by $\tilde{\gamma}_1$, $\tilde{\gamma}_2$, and by $2g_1 - \displaystyle\sum_{p_{i_k} \in X_1} m_{i_k}$ cycles homologous to $\tilde{\lambda}$. Thus the action induced by $T_{\gamma_1} \circ T_{\gamma_2}$ on the set of rth roots of $\mathcal{L}$ is translation by the cocycle that takes B_i to $\left(2g_1 - \displaystyle\sum_{p_{i_k} \in X_1} m_{i_k} \right) \cdot <B_i, \gamma_1>$, i.e., it
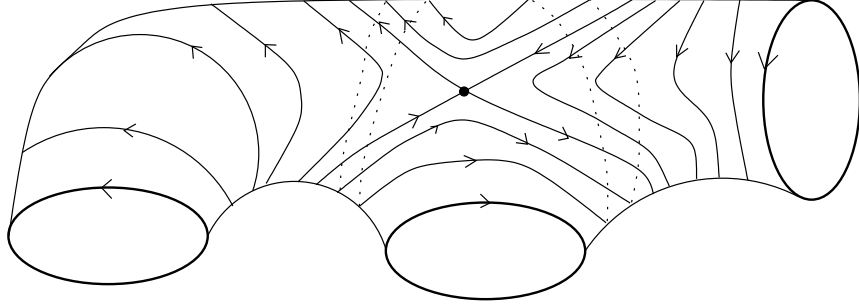
FIGURE 3. A singular vector field, which when joined to $2g_1$ other such pieces, defines a singular vector field in TX^o, homeomorphic to X_1. Removing a small disc near each singularity gives a 2-chain in $\mathcal{L}^o$ with boundary $\tilde{\gamma}_1 + \tilde{\gamma}_2$ plus $2g_1 - \sum_{p_{i_k} \in X_1} m_{i_k}$ 1-cycles homologous to $\tilde{\lambda}$.

is $\left(2g_1 - \sum_{p_{i_k} \in X_1} m_{i_k} \right)$ times the Poincaré dual of γ_1. Note that this is well-defined because the sum $2g_1 + 2g_2 - \sum_i m_i = 2g - 2 - \sum m_i$ is divisible by r.

It is easy to see that for any basis cycle B_i, for any integers g_1 and g_2 that sum to $g - 1$, and for any subset $\{p_{i_k}\}$ of the marked points, there is a bounding pair γ_1 and γ_2 that are each dual to B_i, and which cut X into two surfaces X_1 and X_2 of genera g_1 and g_2 respectively. Moreover, this can be done so that X_1 contains the points $\{p_{i_k}\}$, and X_2 contains the remaining marked points. Thus the action of $\Gamma_{g,n}$ on the rth roots of $\mathcal{L}$ includes all translations by elements in the subgroup $< 2, m_1, \ldots, m_n > H^1(X, \mathbb{Z}/r\mathbb{Z})$ if $g > 1$ and $< m_1, \ldots, m_n > H^1(X, \mathbb{Z}/r\mathbb{Z})$ if $g = 1$.

Proposition 3.3.8 below shows that the Torelli group is generated by Dehn twists on bounding (cut) pairs, as above, and by Dehn twists on separating simple closed curves (also called bridges). Thus all that remains in the proof of the lemma is to show that Dehn twists along separating simple closed curves act trivially on $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$. But this follows immediately from the generalized Picard-Lefschetz formula (2) and the fact that the intersection of a separating simple closed curve with the B_i is zero. This completes the proof of the lemma and also the proof of Theorem 3.3.5. □

The following proposition gives a set of generators for the Torelli group in all genera greater than zero, and for all n. Although this result follows easily from known results, it does not seem to exist, as such, in the literature, so we include it here for completeness.

Proposition 3.3.8. *For $g \geq 1$ and $n \geq 0$ the Torelli group $\mathcal{I}_{g,n}$ is generated by Dehn twists on bounding pairs of simple closed curves and Dehn twists on separating simple closed curves. If $g \neq 2$, a finite subset of such twists generate $\mathcal{I}_{g,n}$*

Proof. In the case that $g \geq 3$ and $n = 0$ the result is due to Powell [28], and Johnson [17] proved that a finite subset of these will suffice to generate $\mathcal{I}_{g,0}$. In the case of

$g = 1$ and $n = 0$ or $n = 1$ it is a classical fact that $\mathcal{I}_{1,0} = \mathcal{I}_{1,1} = \{1\}$. And for $g = 2$ and $n = 0$ the result of the proposition was proved by G. Mess [25], but no finite set of generators is known.

For every $n \geq 1$ Birman [4, pp158-60] gives a finite collection of bounding pairs of simple closed curves (unlike those of Johnson, these bounding pairs may cut the surface so that one of the resulting pieces has genus zero) such that Dehn twists on these generate the kernel $K_{g,n}$ of the natural homomorphism $i_* : \Gamma_{g,n} \longrightarrow \Gamma_{g,n-1}$, and since i_* is surjective, we have an exact sequence

$$1 \longrightarrow K_{g,n} \longrightarrow \mathcal{I}_{g,n} \longrightarrow \mathcal{I}_{g,n-1} \longrightarrow 1.$$

By induction, the Proposition holds for all n.   □

Note 3.3.9. For $g \geq 2$ the number of elements in $\mathcal{S}_{g,n}^{1/r,\mathbf{m}^{even}}[X]$ and $\mathcal{S}_{g,n}^{1/r,\mathbf{m}^{odd}}[X]$ is easily computed as the number of even, respectively odd, theta-characteristics times the degree of the morphism $[r/\ell_{g,r}(\mathbf{m})]$ (which is the order of the group $2 \cdot \mathrm{Jac}_r X$). And it is known [6] that of the 2^{2g} square roots of ω_X, $2^{g-1}(2^g + 1)$ are even, and $2^{g-1}(2^g - 1)$ are odd; so there are $r^{2g}(1/2 + 1/2^{g+1})$ even elements in $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$ and $r^{2g}(1/2 - 1/2^{g+1})$ odd, (Some of these elements may be identified by automorphisms of X).

Similarly, for $g = 1$ the number of elements in the component of $\mathcal{S}_{1,n}^{1/r,\mathbf{0}}$ indexed by d is well-known [13, 11§3 Prop 3.6 (2)], and easily calculated to be $(r/d)^2 \prod_{p|(r/d)}(1 - 1/p^2)$ when X has no automorphisms (generically when $n \geq 2$). And the degree of $[r/\ell_{1,r}(\mathbf{m})]$ is $(r/\ell_{1,r}(\mathbf{m}))^2$.

Note 3.3.10. Using the techniques of the proof of Lemma 3.3.6 to calculate the action of various Dehn twists on $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$, it is not hard to see that, when $r > 2$, the stabilizer of one point in $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$ is not normal in $\Gamma_{g,n}$. That is, the finite cover $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}} \to \overline{\mathcal{M}}_{g,n}$ is not regular. This is an interesting contrast to the fact that the subgroup of $\Gamma_{g,n}$ which fixes *all* the points of $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}[X]$ is normal [31, Theorem C].

3.3.2. *Irreducibility over General Ground Fields.* Although in the previous section we have been working over the complex numbers $\mathbb{C}$, r-spin curves form a smooth Deligne-Mumford stack over $\mathbb{Z}[1/r]$. Moreover, as in the case of $\overline{\mathfrak{M}}_g$ (see [7, §5]), irreducibility in characteristic 0 gives the same result in any characteristic relatively prime to r.

Theorem 3.3.11. *For any field k with characteristic prime to r, the moduli space $\mathcal{S}_{g,n}^{1/r,\mathbf{m}}$ and and its compactification $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$ of pure spin curves over k are irreducible if $\ell_{g,r}(\mathbf{m}) = 1$. In general, over any algebraically closed field, the moduli space is the disjoint union of $d_{g,r}(\mathbf{m})$ irreducible components.*

Proof. The proof is essentially the same as for the case of $\mathcal{M}_g$, but we will sketch the main steps.

First, note that it is well-known and straightforward to prove [32, Lemma 2.3] that an algebraic stack is irreducible or connected if and only if its coarse moduli space has the same property. It is easy to see from the universal deformation of stable spin curves that the stack $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ of smooth spin curves is an open dense

substack of the stack $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ of stable spin curves, and thus its irreducible components are the non-empty intersections of $\mathfrak{S}_{g,n}^{1/r,\mathbf{m}}$ with the irreducible components of $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ [7, Prop 4.15]. So it suffices to prove the theorem in the case of $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$.

Since $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ is smooth, its connected components are irreducible [7, Prop 4.16]. Moreover, if for each s in Spec $\mathbb{Z}[1/r]$, we define $n(s)$ to be the number of connected components of the geometric fibre of $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$ over s, then $n(s)$ is a constant function [7, Prop 4.17].

And finally, by Theorem 3.3.5 the assertion of Theorem 3.3.11 holds over $\mathbb{C}$ for the moduli space $\overline{\mathcal{S}}_{g,n}^{1/r,\mathbf{m}}$, and thus for the stack $\overline{\mathfrak{S}}_{g,n}^{1/r,\mathbf{m}}$. Consequently the assertion must hold for both the stack and its moduli space over all algebraically closed fields. And in the case that $\ell_{g,r}(\mathbf{m}) = 1$ the stack and its moduli space must be irreducible over any field.   □

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DEPARTMENT OF MATHEMATICS, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602

E-mail address: `jarvis@math.byu.edu`